'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2023

CC2-MATHEMATICS

ALGEBRA

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) If $p \geq q \geq 5$ and $p,\ q$ are both prime, then prove that $p^2 - q^2$ is divisible by 24.

(b) Find $g \circ f$ if $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = |x| + x$, $x \in \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = |x| - x$, $x \in \mathbb{R}$.

(c) Use Euclidean algorithm to find the integer x and y such that

$$\gcd(72,\ 92) = 72x + 92y$$

(d) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 7 & 2 & 9 & 6 & 5 & 4 & 1 \end{pmatrix}$. Express α as a product of 2-cycles. Is α an even permutation? Also find α^{-1}.

(e) If A and B are two square matrices of the same order and A is non-singular, prove that B and ABA^{-1} have same eigen values.

(f) Find the principal value of $(-1+i)^{1+i}$.

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) State Sturm's theorem. Using Strum's function to separate the roots of the equation

$$x^3 + x^2 - 2x - 1 = 0$$

(b) If x is real prove that

$$i \log \frac{x-i}{x+i} = \pi - 2\tan^{-1} x\ , \quad \text{if } x > 0$$

$$= -\pi - 2\tan^{-1} x\ , \quad \text{if } x \leq 0$$

(c) If a, b, c, d be all positive real numbers and $s = a + b + c + d$, prove that

$$81\, abc \leq (s-a)(s-b)(s-c)(s-d) \leq \frac{81}{256} s^4$$

(d) Find the inverse of the matrix using Cayley-Hamilton theorem

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

(e) (i) Solve $15x \equiv 9 \pmod{18}$. 3+3

(ii) Prove that $3 \cdot 4^{n+1} \equiv 3 \pmod 9$ for all positive integers n.

(f) Solve by Ferrari's method $x^4 + 4x^3 - 6x^2 + 20x + 8 = 0$. 6

GROUP-C

3. Answer any ***two*** questions: 12×2 = 24

(a) (i) If α, β, γ be the roots of the equation $x^3 + qx + r = 0$, find the equation whose roots are $\frac{\alpha+\beta}{\alpha\beta}$, $\frac{\beta+\gamma}{\beta\gamma}$, $\frac{\gamma+\alpha}{\gamma\alpha}$ and hence find out the value of $\sum\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{\gamma}\right)$. 4+1

(ii) Obtain the row reduced echelon form of the matrix: 3+1

$$\begin{pmatrix} 1 & 3 & -2 & -3 \\ 2 & 1 & 4 & -1 \\ 3 & 2 & 1 & 5 \\ 1 & 3 & 5 & 4 \end{pmatrix}$$

and find the rank.

(iii) Investigate the nature of roots of the equation 3

$$x^5 - x^4 + 8x^2 - 9x - 15 = 0$$

by using Descartes' rule of signs.

(b) (i) Determine the conditions of λ and μ for which the system of equations 6

$$-4x + 2y - 9z = 2$$
$$3x + 4y + z = 5$$
$$3x + 4y + \lambda z = \mu$$

admits of (I) only one solution
(II) no solution
(III) many solutions.

(ii) State Cauchy-Schwartz inequality for real numbers. 2

(iii) Prove that $n! > 2^n$ for all $n \geq 4$. 4

(c) (i) Let R be a relation on a set A. Define $\tau(R) = R \cup R^{-1} \cup \{(x, x)/x \in A\}$. Show that R is reflexive and symmetric. 4

(ii) If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, prove that 5

$$\sum\cos^2\alpha = \sum\sin^2\alpha = \frac{3}{2}$$

(iii) If α, β, γ be the roots of equation $x^3 + px^2 + qx + r = 0$ then find the value of $\sum(\beta+\gamma-\alpha)^3$. 3

(d) (i) For the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, find a matrix P such that $P^{-1}AP$ is a diagonal matrix. 6

(ii) Solve the equation $x^4 - 12x^3 + 48x^2 - 72x + 35 = 0$ by removing the second term. 4

(iii) Stage Fundamental theorem of arithmetic. 2

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