'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2023

CC5-MATHEMATICS

THEORY OF REAL FUNCTIONS AND INTRODUCTION TO METRIC SPACE

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours — Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: $3\times 4 = 12$

(a) Evaluate $\lim\limits_{x\to 3}\left([x]-\left[\dfrac{x}{3}\right]\right)$. 3

(b) If a function $f:[0,1]\to\mathbb{R}$ is continuous on $[0,1]$ and f assumes only rational values on $[0,1]$, then prove that f is a constant. 3

(c) Discuss the applicability of M.V.T. for $f(x)=|x|$ in $[-1,\ 1]$ and $[0,1]$. 3

(d) Is the function $f(x)=\dfrac{x}{x+1}$ uniformly continuous for $x\in[0,2]$? Justify your answer. 3

(e) Prove that the intersection of two open sets in a metric space is open. 3

(f) For a subset A of a metric space $(X,\ d)$, show that $\text{Int}(A)=X-\text{cl}(X-A)$. 3

GROUP-B

2. Answer any ***four*** questions: $6\times 4 = 24$

(a) If $f:[a,b]\to\mathbb{R}$ is continuous, then show that f attains its supremum and infimum on $[a,b]$. 6

(b) Suppose $f(x)$ is a function satisfying the conditions: 6

(i) $f(0)=2$, $f(1)=1$

(ii) f has a minimum value at $x=\dfrac{1}{2}$

(iii) $f'(x)=2ax+b$ for all x.

Determine the constants a, b and the function $f(x)$.

(c) State and prove Rolle's Theorem. 6

(d) If $f:(a,b)\to\mathbb{R}$ is continuous, then prove that f is uniformly continuous on (a,b) iff $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exists finitely. 6

(e) For a metric space (X,d), show that $d^*(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$, $\forall x,y\in X$ is a bounded metric on X. 6

(f) If $\{x_n\}$ and $\{y_n\}$ are two sequences in a metric space (X,d) such that $x_n\to x$ and $y_n\to y$, then prove that $d(x_n,y_n)\to d(x,y)$. 6

GROUP-C

3. Answer any ***two*** questions: 12×2 = 24

(a) (i) Suppose $g:D\to\mathbb{R}$ be a function on $D\subseteq\mathbb{R}$ and $(c,\infty)\subset D$ for some $c\in\mathbb{R}$. Prove that $\lim_{x\to\infty} g(x)=l$, $l\in\mathbb{R}$ iff $\lim_{x\to 0^+} g\left(\dfrac{1}{x}\right)=l$. 6

(ii) Using Cauchy's principle prove that $\lim_{x\to 0}\cos\dfrac{1}{x}$ does not exist. 6

(b) (i) A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ satisfies $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$ and f is continuous at 0. Prove that f is continuous at every $c\in\mathbb{R}$. Also deduce that $f(x)=kx$ for all $x\in\mathbb{R}$ and for some $k\in\mathbb{R}$. 6

(ii) A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is defined by $f(x)=x$, $x\in\mathbb{Q}$; $f(x)=0$, $x\in\mathbb{R}\setminus\mathbb{Q}$. Show that f is continuous at 0 and f has a discontinuity of the 2nd kind at every other point in $\mathbb{R}$. 6

(c) (i) Suppose a function $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and $f'(x)\neq 0$ on (a,b). If $f(a)$ and $f(b)$ are of opposite signs, show that there is a unique $x_0\in(a,b)$ such that $f(x_0)=0$. 6

(ii) Find the global maximum and the global minimum of the function f on $\mathbb{R}$, where $f(x)=\dfrac{x^2-2x+4}{x^2+2x+4}$, $x\in\mathbb{R}$. 6

(d) (i) Consider the space of all sequences of complex numbers and define the function d on X by 6

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{|x_n-y_n|}{(1+|x_n-y_n|)},\ \forall x=\{x_n\},\ y=\{y_n\}\in X$$

Show that (X,d) is a metric space.

(ii) Suppose X be the set of all continuous real valued functions defined on $[0,1]$ and $d(x,y)=\int_0^1|x(t)-y(t)|\,dt$, $x,\ y\in X$. Show that (X,d) is not complete. 6

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