'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2023

CC6-MATHEMATICS

GROUP THEORY-I

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) List all even permutations of S_4.

(b) Give an example of a non-cyclic group each of whose proper subgroups is cyclic.

(c) Let $G = S_3$ and $G' = \{1, -1\}$ and $\varphi : G \to G'$ is defined by

$$\varphi(x) = \begin{cases} 1 \ , & \text{if } x \text{ is an even permutation} \\ -1 \ , & \text{if } x \text{ is an odd permutation} \end{cases}$$

then determine $\ker\varphi$.

(d) Find the center of the symmetric group S_3.

(e) If in a group G, $(a*b)^{-1} = a^{-1}*b^{-1}$ for all $a, b \in G$, then show that G is a commutative group.

(f) Find the number of generators of the group $(\mathbb{Z}_{15}, +)$.

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) Prove that every subgroup of a cyclic group is cyclic. 6

(b) Let H and K be two subgroups of a group G. Then show that HK is a subgroup of G if and only if $HK = KH$. 6

(c) (i) Show that A_4 has no subgroup of order 6. 4

(ii) Let G be a group of order 28. Show that G has a non-trivial subgroup. 2

(d) (i) Let H be a subgroup of a group G. Define $N(H)=\{g\in G \mid gHg^{-1}=H\}$. Show that $N(H)$ is a subgroup of G. Find $N(H)$ if H is normal in G. 2+2

(ii) Prove that every group of prime order is cyclic. 2

(e) Prove that every finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$. 6

(f) Let $\alpha=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 2 & 1 & 5 & 4\end{pmatrix}$ and $\beta=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 4 & 3 & 5 & 1\end{pmatrix}$ in S_5. Find a permutation γ in S_5 such that $\alpha\gamma=\beta$. 6

GROUP-C

3. Answer any ***two*** questions: $12\times 2=24$

(a) (i) Prove that in a cyclic group of even order, there is exactly one element of order 2. 3

(ii) Let $G=\langle a\rangle$ be an infinite cyclic group. Show that G has only two generators. 3

(iii) Prove that the group $4\mathbb{Z}/12\mathbb{Z}\simeq\mathbb{Z}_3$. 3

(iv) Find all normal subgroups of S_4. 3

(b) (i) Let G be a group of order 15 and A and B are subgroups of G of order 5 and 3, respectively. Show that $G=AB$. 4

(ii) Prove that a finite semigroup $(S,*)$ is a group if and only if $(S,*)$ satisfies the cancellation laws (i.e., $a*c=b*c$ implies $a=b$ and $c*a=c*b$ implies $a=b$ for all $a,b,c\in S$). 6

(iii) State second isomorphism theorem for groups. 2

(c) (i) Let H be a subgroup of a group G. Prove that any two left cosets of H in G are either identical or they have no common element. 4

(ii) Let H be a subgroup of a group G. If $x^2\in H$ for all $x\in G$, then prove that H is a normal subgroup of G and G/H is commutative. 5

(iii) Find all subgroups of $\mathbb{Z}/21\mathbb{Z}$. 3

(d) (i) Show that every group of order 14 contains only 6 elements of order 7. 4

(ii) Let X be a non-empty set and $P(X)$ be the power set of X. Examine if $P(X)$ is a group under the composition $*$ defined by 6

$$A*B=A\Delta B=(A\setminus B)\cup(B\setminus A)\ ,\quad \forall\, A,B\in P(X)$$

(iii) Show that $(\mathbb{Z},+)$ and $(\mathbb{Q},+)$ are not isomorphic. 2

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