'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2023

CC7-MATHEMATICS

RIEMANN INTEGRATION AND SERIES OF FUNCTIONS

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: $3\times 4 = 12$

(a) Examine the uniform convergence of the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, where 3

$$f_n(x)=\frac{nx}{1+n^2x^2}\quad,\quad x\geq 0$$

(b) A function f is defined by $f(x)=\sum\limits_{n=1}^{\infty}\frac{\cos nx}{10^n}\ ,\ \ x\in\mathbb{R}$ 3

Show that f is continuous for any $x\in\mathbb{R}$.

(c) Prove that, $\mathcal{B}(x,\ y)=2\int\limits_0^{\pi/2}(\sin t)^{2x-1}(\cos t)^{2y-1}\,dt$, 3

where $\mathcal{B}$ represents beta function.

(d) Examine the convergence of $\int\limits_1^{\infty}\frac{dx}{(1+x)\sqrt{x}}$. 3

(e) Find the Fourier coefficients for the function $f(x)=|x|$, in $-\pi\leq x\leq\pi$. 3

(f) Evaluate the integral $\int\limits_{-2}^{2}([x^2]+|x|)\,dx$, where $[x]$ = greatest integer $\leq x$. 3

GROUP-B

Answer any *four* questions $6\times 4 = 24$

2. Test the convergence of the improper integral $\int\limits_0^{\infty}\frac{\sin x^m}{x^n}\,dx$. 6

3. Expand the periodic function $f(x)=x^2,\ 0\leq x\leq l$ of period l, in a series of cosines only and hence deduce that 6

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots=\frac{\pi^2}{6}$$

4. Assuming the power series expansion for $\frac{1}{\sqrt{1-x^2}}$ as 6

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{x^2}{2}+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\cdots\cdots,$$

Obtain the power series expansion for $\sin^{-1}x$ and deduce that

$$1+\frac{1}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}+\cdots\cdots=\frac{\pi}{2}$$

5. Let $D\subseteq\mathbb{R}$ and $\forall n\in\mathbb{N}$, $f_n:D\to\mathbb{R}$ be continuous functions. If the sequence $\{f_n\}$ be uniformly convergent on D to a function f, then prove that f is continuous on D. 6

6. Show that $\Gamma\left(n+\frac{1}{2}\right)=\dfrac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2^n}\,\Gamma(n+1)}$ 6

7. State and prove Riemann-Lebesgue Lemma. 6

GROUP-C

Answer any *two* questions $12\times 2=24$

8. (a) Show that $\displaystyle\int_0^\infty \frac{\sin ax}{x}\,dx=\frac{\pi}{2}$ or 0 or $-\dfrac{\pi}{2}$ according as a is positive or zero or negative. 6

(b) Prove that $\displaystyle\int_0^\infty \frac{x^{s-1}}{1+x^t}\,dx$ is convergent iff $0<s<t$. 6

9. (a) Starting from the power series expansion of $\dfrac{1}{1+x^2}$ with proper justification, show that 4+2

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\cdots\qquad(-1\le x\le 1)$$

Hence deduce that $\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots\cdots$

(b) Show that, when $0<x<\pi$ 6

$$\pi-x=\frac{1}{2}\pi+\frac{\sin 2x}{1}+\frac{\sin 4x}{2}+\frac{\sin 6x}{3}+\cdots\cdots$$

10.(a) Prove that a bounded function f is integrable in $[a,\,b]$ if the set of its points of discontinuity has a finite number of limit points. 6

(b) Define f on $[a,\,b]$ as follows: 6

$$f(x)=\begin{cases}1/q^2 &,\quad \text{when } x=\dfrac{p}{q}\\[2ex] 1/q^3 &,\quad \text{when } x=\sqrt{\dfrac{p}{q}}\end{cases}$$

where $p,\,q$ are relatively prime integers and $f(x)=0$ elsewhere, then show that f is Riemann integrable on $[a,\,b]$.

11.(a) Let $f_n(x)=\dfrac{nx}{1+n^2x^2}-\dfrac{(n-1)x}{1+(n-1)^2x^2}$, $x\in[0,\,1]$. 6

Show that at $x=0$,

$$\frac{d}{dx}\sum f_n(x)\neq\sum\frac{d}{dx}f_n(x)$$

(b) Find the Fourier series of the periodic function f with period 2π defined as follows: 6

$$f(x)=\begin{cases}0 &,\quad \text{for } -\pi<x\le 0\\ x &,\quad \text{for } 0\le x\le\pi\end{cases}$$

What is the sum of the series at $x=0,\ \pm\pi,\ 4\pi,\ -5\pi$?

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