'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2022

CC2-MATHEMATICS

ALGEBRA

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: $3\times 4=12$

(a) If $T:\mathbb{R}^3\to\mathbb{R}$ and $T(x_1,x_2,x_3)=x_1^2+x_2^2+x_3^2$ then show that T is not a Linear Transformation. 3

(b) If a, b, x are real and $\operatorname{mod}(a+ib)=1$, prove that $(a+ib)^{ix}$ is purely real. 3

(c) A relation ρ on $\mathbb{Z}$ is defined by $\rho=\{(a,b)\in\mathbb{Z}\times\mathbb{Z}: a-b$ is divisible by 7$\}$. Show that ρ is an equivalence relation. 3

(d) If $x^3+3px+q$ has a factor of the form $(x-\alpha)^2$, then show that $q^2+3p^3=0$. 3

(e) Prove that $n(n+1)^2>4(n!)^{3/n}$ where n be a positive integer greater than 1. 3

(f) Determine the rank of the matrix 3

$$\begin{pmatrix}1&2&1&0\\2&4&8&6\\3&6&6&3\end{pmatrix}.$$

GROUP-B

2. Answer any ***four*** questions: $6\times 4=24$

(a) Obtain the fully reduced normal form of the matrix 6

$$\begin{pmatrix}0&0&1&2&1\\1&3&1&0&3\\2&6&4&2&8\\3&9&4&2&10\end{pmatrix}$$

(b) If $\log\sin(x+iy)=u+iv$ $(0<x<\pi)$, prove that 6

(i) $u=\dfrac{1}{2}\log(\cosh^2 y-\cos^2 x)$

(ii) $v=\tan^{-1}(\cot x\tanh y)$

(c) If α be a non-real root of $x^7=1$, find the equation whose roots are 6

$$(\alpha+\alpha^6),\ (\alpha^2+\alpha^5),\ (\alpha^3+\alpha^4).$$

(d) Find the eigenvalues and the corresponding eigenvectors of the following real matrix. 6

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 1 \\ 3 & 5 & 6 \end{pmatrix}$$

(e) If a, b, c, d are positive and not all equal then prove that 6

$$\frac{3}{b+c+d}+\frac{3}{c+d+a}+\frac{3}{d+a+b}+\frac{3}{a+b+c}>\frac{16}{a+b+c+d}.$$

(f) (i) Find the least positive residues in $3^{36}\ (\text{mod } 77)$. 3+3

(ii) If a mapping $f: A \to \mathbb{R}$, where $A=\{x \mid 0<x<1\}$ is defined by $f(x)=\dfrac{2x-1}{1-|2x-1|}$, $x \in A$ then show that f is bijective.

GROUP-C

3. Answer any ***two*** questions: $12\times 2 = 24$

(a) (i) State and prove Division algorithm. 6

(ii) The matrix of a Linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the ordered basis $\{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$ of $\mathbb{R}^3$ is $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix}$. Find $T(x, y, z)$, where $(x, y, z) \in \mathbb{R}^3$. Is T invertible? 6

(b) (i) Solve the biquadratic equation by Ferrari's method: 6

$$x^4+2x^3-7x^2-8x+12=0.$$

(ii) Prove that the product of any m consecutive integers is divisible by m. 3

(iii) Find all values of $(-i)^{3/4}$. 3

(c) (i) If the roots α, β, γ of the equation $x^3+qx+r=0$ are in A.P., show that the rank of the matrix $\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{pmatrix}$ is 2. 4

(ii) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are n distinct roots of the equation $x^n-1=0$, then prove that $(a+b\alpha_1)(a+b\alpha_2)\cdots\cdots(a+b\alpha_n)=a^n+(-1)^{n-1}b^n$. 5

(iii) Prove that $3.4^{n+1} \equiv 3\ (\text{mod } 9)$ for all positive integer n. 3

(d) (i) Determine the conditions for which of the following system of linear equations 6

$$x+2y+z=1$$
$$2x+y+3z=b$$
$$x+ay+3z=b+1$$

has (A) Unique solution, (B) No solution and (C) many solutions.

(ii) Find the inverse of the given matrix A by using Cayley Hamilton theorem: 6

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$$

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