'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2022

GE1-P1-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

The question paper contains GE1, GE2, GE3, GE4 and GE5. Candidates are required to answer any *one* from the *five* courses and they should mention it clearly on the Answer Book.

GE1

CALCULUS, GEOMETRY AND DIFFERENTIAL EQUATION

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Evaluate $\lim\limits_{x\to 0}\dfrac{e^{3x}-3x-1}{1-\cos x}$. 3

(b) Find the equation of the tangent line to the curve $x=2t+4$, $y=8t^2-2t+4$ at $t=2$ without eliminating the parameter. 3

(c) Find the envelope of the family of ellipses $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where the two parameters a, b are connected by the relation $a+b=c$, c is a constant. 3

(d) If the straight line $r\cos(\theta-\alpha)=p$ touches the parabola $\dfrac{l}{r}=1+\cos\theta$, show that $p=\dfrac{l\sec\alpha}{2}$. 3

(e) Find the degree of the differential equation satisfying 3

$$\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y).$$

(f) Find the nature of the conic: 3

$$x^2-5xy+y^2+8x-20y+15=0$$

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) If $y=(\sqrt{1+x^2}+x)^m$, prove that $(1+x^2)y_2+xy_1-m^2y=0$. 6

(b) Find the asymptotes of the curve: 6

$$x^3+x^2y-xy^2-y^3+x^2-y^2=2$$

(c) Find the area in the first quadrant included between the parabola $y^2=bx$ and the circle $x^2+y^2=2bx$. 6

(d) Solve $(x+\tan y)dy=\sin 2y\,dx$. 6

(e) Reduce the equation $5x^2-6xy+5y^2-4x-4y-4=0$ to its canonical form and hence find the nature of the conic. 6

(f) Find the equation of the sphere which passes through the points $(0,-2,4)$ and $(2,-1,-1)$ and whose centre lies on the straight line $5y+2z=0=2x-3y$. 6

GROUP-C

Answer any *two* questions from the following 12×2 = 24

3. (a) Find the values of p and q such that 6

$$\lim_{x\to 0}\frac{x(1-p\cos x)+q\sin x}{x^3}=\frac{1}{3}$$

(b) Find the range of values of x for which $y=x^4-6x^3+12x^2+5x+7$ is concave upwards or downwards. Find also its point of inflexion, if any. 6

4. (a) Solve: $\frac{dy}{dx}+(x\tan^{-1}y-x^3)(1+y^2)=0$, when $x=0$, $y=1$. 6

(b) Solve: $\frac{dy}{dx}=\frac{y-x+1}{y+x+5}$. 6

5. (a) Find the smallest sphere which touches the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z-6}{1}$ and $\frac{x+3}{7}=\frac{y+3}{-6}=\frac{z+3}{1}$. 6

(b) Find the point of inflexion on the curve $r=\frac{ae^{\theta}}{(1+\theta)}$. 6

6. (a) The origin is shifted to the point $(3,-1)$ and the axes are rotated through an angle $\tan^{-1}\frac{3}{4}$. If the coordinate of a point is $(5,10)$ in the new system, find the coordinate in the old system. 6

(b) Find the area of the region bounded by the upper half of the circle $x^2+y^2=25$, the x-axis and the ordinates $x=-3$ and $x=4$. 6

GE2

ALGEBRA

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) If α,β,γ be the roots of the equation $2x^3-x^2-18x+9=0$, then find the value of $\sum\alpha^2\beta^2$. 3

(b) $x+\frac{1}{x}=2\cos\frac{\pi}{7}$, then find the value of $x^7+\frac{1}{x^7}$. 3

(c) If $A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, find the rank of $A+A^2$. 3

(d) Examine whether $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,\ y,\ z) = (x+2y+3z,\ 3x+2y+z,\ x+y+z)$; $(x,\ y,\ z) \in \mathbb{R}^3$ is a linear mapping. 3

(e) Examine whether the relation ρ on $\mathbb{Z}$ defined by 3

$\rho = \{(a,\ b) \in \mathbb{Z}\times\mathbb{Z} : (3a+4b) \text{ is divisible by } 7\}$ is an equivalence relation or not.

(f) If λ is an eigenvalue of a non-singular matrix A, show that λ^{-1} is an eigenvalue of A^{-1}. 3

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) Reduce the equation $x^3 - 3x^2 + 12x + 16 = 0$ to its standard form and then solve the equation by Cardan's method. 6

(b) Using Euclidean algorithm, find the values of U and V, such that 6

$$1269U + 297V = 135$$

(c) If $\alpha,\ \beta,\ \gamma$ are the roots of the equation $x^3 + 3x + 1 = 0$, find the equation whose roots are, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$, $\frac{\beta}{\gamma}+\frac{\gamma}{\beta}$ and $\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}$. Hence find the value of $\sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$. 5+1

(d) If $x = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ and $\theta \in \mathbb{R}$, then prove that $\theta = -i\log\tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)$. 6

(e) Convert the matrix 6

$$\begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix}$$

into echelon form, and hence find the solution set of the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 &= 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 &= 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 &= 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 &= 14 \end{aligned}$$

(f) Let $a,\ b,\ c$ be positive real numbers and $a+b+c=1$. Show that 6

$$8abc \le (1-a)(1-b)(1-c) \le \frac{8}{27}.$$

GROUP-C

Answer any *two* questions 12×2 = 24

3. (a) Using the theory of congruences to prove that $7 \mid 2^{5n+3} + 5^{2n+3},\ \forall n \ge 1$. 3

(b) If $ax \equiv ay \pmod{m}$ and $\gcd(a,\ m) = 1$, show that $x \equiv y \pmod{m}$. 3

(c) If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then prove that for every integer $n(\ge 3)$, $A^n = A^{n-2} + A^2 - I$. Also find A^{50}. 4+2

4. (a) For any two integers m and n with $n \neq 0$, show that there exists unique integers q and r such that $m = nq + r$, where either $r = 0$ or $r < n$. 6

(b) If $\tan \log(x+iy) = \alpha + i\beta$, where $\alpha^2 + \beta^2 \neq 1$, then show that 6

$$\tan \log(x^2 + y^2) = \frac{2\alpha}{1-\alpha^2-\beta^2}$$

5. (a) If p and p^2+8 are both prime numbers then find the value of p. 4

(b) For what values of k and μ the following system of equations: $x_1 + 4x_2 + 2x_3 = 1$; $2x_1 + 7x_2 + 5x_3 = 2k$; $4x_1 + \mu x_2 + 10x_3 = 2k+1$, has (i) a unique solution, (ii) no solution, (iii) infinitely many solutions over the field $\mathbb{Q}$. 8

6. (a) Solve the equation $x^4 + 6x^3 + 18x^2 + 24x + 16 = 0$ 4

(b) Determine the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) of $\mathbb{R}^3$ to the vectors (1, 1), (2, 3), (3, 2) respectively. 8

(i) Find $\ker T$ and $\operatorname{Im} T$.

(ii) Prove that T is not one-to-one but onto.

GE3

DIFFERENTIAL EQUATION AND VECTOR CALCULUS

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) If S is defined by the rectangle $|x| \leq a$, $|y| \leq b$, show that the function $f(x, y) = x^2 + y^2$ satisfies the Lipschitz condition. Find the Lipschitz constant. 3

(b) Find the Wronskian of $\{1, \sin x, \cos x\}$. 3

(c) If $\vec{r} = 3t^3\vec{i} + 2t^2\hat{j} + t^3\hat{k}$ then find $\frac{d^2\vec{r}}{dt^2} \times \frac{d\vec{r}}{dt}$. 3

(d) Find a particular integral of $(D^2+4)y = \sin 2x$. 3

(e) If $|\vec{a}| = 1$, $|\vec{b}| = 1$, $|\vec{c}| = 2$ and $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = 0$, find the angle between $\vec{a}$ and $\vec{c}$. 3

(f) Evaluate $\frac{1}{D^2-1} 4xe^x$. 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) (i) Solve: $\frac{d^4y}{dx^4} + y = \cosh 4x \sinh 3x$. 3

(ii) Solve by the method of variation of parameters: $\frac{d^2y}{dx^2} + y = 4\sin x$. 3

(b) (i) Solve by the method of undetermined coefficients: 4

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 12e^x - 34\sin 2x$$

(ii) Test the continuity at $t=0$ of 2

$$\vec{r}(t)=\begin{cases} t^2\hat{i}+\dfrac{\sin t}{t}\hat{j}+t\hat{k} & ; \quad t\neq 0 \\ j & ; \quad t=0 \end{cases}$$

(c) (i) Show that $\vec{\nabla}\times\{\vec{r}\,f(\vec{r})\}=0$ where $r=|\vec{r}\,|$. 3

(ii) A particle moves along the curve $x=2t^2$, $y=t^2-4t, z=-t-5$, where t is the time. Find the components of its velocity and acceleration at time $t=1$ in the direction $\hat{i}-2\hat{j}+2\hat{k}$. 3

(d) (i) If $\vec{r}=a\cos t\hat{i}+a\sin t\hat{j}+at\tan\alpha\hat{k}$, then prove that $\left[\dfrac{d\vec{r}}{dt}, \dfrac{d^2\vec{r}}{dt^2}, \dfrac{d^3\vec{r}}{dt^3}\right]=a^3\tan\alpha$. 3

(ii) If $\vec{A}=(3x^2+6y)\hat{i}-14yz\hat{j}+20xz^2\hat{k}$, then evaluate $\int_C \vec{A}\cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the path $x=t$, $y=t^2$, $z=t^3$. 3

(e) Solve: $\dfrac{d^2y}{dx^2}-y=\dfrac{2}{1+e^x}$. 6

(f) Solve the system of simultaneous equations: 6

$$\frac{dy}{dx}+2y-3z=x$$

$$\frac{dz}{dx}+2z-3y=e^{2x}$$

GROUP-C

3. Answer any ***two*** questions from the following: 12×2 = 24

(a) (i) Solve: $(1+2x)^2\dfrac{d^2y}{dx^2}-6(1+2x)\dfrac{dy}{dx}+16y=8(1+2x)$. 6

(ii) Solve: $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=\dfrac{e^{-x}}{x^2}$. 6

(b) (i) Solve: $\dfrac{d^2y}{dx^2}+(x-1)^2\dfrac{dy}{dx}-4(x-1)y=0$ in series about the ordinary point $x=1$. 6

(ii) Solve the differential equation: 6

$$\frac{d^2y}{dx^2}+a^2y=\sec ax$$

(c) (i) If $\vec{a}$ and $\vec{b}$ are constant vectors prove that $\text{div}\{(\vec{r}\times\vec{a})\times\vec{b}\}=-2(\vec{a}\cdot\vec{b})$ and $\text{curl}\{(\vec{r}\times\vec{a})\times\vec{b}\}=\vec{b}\times\vec{a}$, where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $r=|\vec{r}\,|$. 3+3

(ii) A particle moves along the curve $x=2t^2$, $y=t^2-4t$, $z=3t-5$. Find the components of velocity and acceleration at time $t=1$, in the direction $\hat{i}-3\hat{j}+2\hat{k}$. 6

(d) (i) Prove that $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar if and only if $\vec{a},\vec{b},\vec{c}$ are coplanar. 4

(ii) Prove that the necessary and sufficient that the vector $\vec{a}(t)$ have constant direction is $\vec{a}\times\frac{d\vec{a}}{dt}=0$. 4

(iii) Show that the vector $\frac{\vec{r}}{r^3}$ is both solenoidal and irrotational, where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $r=|\vec{r}|$. 4

GE4

GROUP THEORY

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) Prove that a group $(G,\circ)$ is abelian iff $(a\circ b)^{-1}=a^{-1}\circ b^{-1}$ for all $a,b\in G$. 3

(b) Find the order of the permutation $\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 4 & 5 & 1 & 3 & 2 & 8 & 6 & 7\end{pmatrix}$ 3

(c) Prove that if a is a generator of a cyclic group G, then its inverse a' is also a generator of G. 3

(d) Suppose that G is a group and $Z(G)=\{g:gx=xg,\forall x\in G\}$. Prove that $Z(G)$ is a normal subgroup of G. 3

(e) Let G be a group and $a\in G$. If $O(a)$ be infinite and p be a positive integer, then prove that $O(a^p)$ is also infinite. 3

(f) State Second Isomorphism Theorem. 3

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) (i) Let H and K be finite subgroups of a group G such that HK is a subgroup of G. Show that 4+2

$$O(HK)=\frac{O(H)\cdot O(K)}{O(H\cap K)}$$

(ii) Find all cyclic subgroups of Klein's 4 group.

(b) (i) In a group $(G,\cdot)$, $a^{n+1}b^{n+1}=b^{n+1}a^{n+1}$ and $a^nb^n=b^na^n$ hold for all $a,b\in G$ and for some integer n. Prove that G is abelian. 4+2

(ii) Find all elements of order 8 in the group $(\mathbb{Z}_{24},+)$

(c) For $n>1$, prove that the number of ever permutations and odd permutations in S_n, are equal. 6

(d) Let $f:G\to G'$ be a group homomorphism. If H' be a subgroup of G' then prove that $f^{-1}(H')=\{a\in G\mid f(a)\in H'\}$ is a subgroup of G. Further prove that, if H' is normal in G', then $f^{-1}(H')$ is normal in G. 3+3

(e) (i) Show that $(\mathbb{Q},+)$ is not cyclic 3+3

(ii) Prove that every proper subgroup of a group of order 6 is cyclic.

(f) (i) Show that the groups $(\mathbb{R}^*,\cdot)$ and $(\mathbb{R},+)$ are not isomorphic. $[\mathbb{R}^*=\mathbb{R}-\{0\}]$. 3+3

(ii) Let G be a group of order 10 and G' be a group of order 6. Prove that there does not exist a homomorphism of G onto G'.

GROUP-C

3. Answer any ***two*** questions: $12\times 2 = 24$

(a) (i) Let G be a finite cyclic group generated by a. Prove that $O(G)=n$ iff $O(a)=n$ 6

(ii) Let H be the set of all real matrices 6

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc=1\right\},\ a, b, c, d \in R.$$

Prove that H forms a non-commutative group under matrix multiplication.

(b) (i) If H be a subgroup of a cyclic group G, then prove that the quotient group G/H is cyclic. Is the converse true? Justify your answer. 6

(ii) Prove that every subgroup of a cyclic group is cyclic. 6

(c) (i) Let $(G, \circ)$ be a group and $a, b \in G$. If $a\circ b = b\circ a$ and $O(a)$ and $O(b)$ are prime to each other, show that $O(a\circ b) = O(a)\circ O(b)$. 6

(ii) Let S be the set of all permutations on the set $\{1, 2, 3\}$. Show that S forms a non-abelian group with respect to multiplication. 6

(d) (i) Find all homomorphisms from the group $(\mathbb{Z}_6, +)$ to $(\mathbb{Z}_4, +)$. 6

(ii) Let H be a cyclic subgroup of a group G. If H is normal in G, then prove that every subgroup of H is normal in G. 6

GE5

NUMERICAL METHODS

GROUP-A

1. Answer any ***four*** questions from the following: $3\times 4 = 12$

(a) Find the absolute, relative and percentage errors when $3/7$ is approximated to 5 significant figures. 3

(b) Find an interpolating polynomial which interpolates $f(x)$ in such a way that $f(0)=1$, $f(1)=0$, $f(2)=2$. 3

(c) Show that $\Delta \log f(x) = \log\left\{1+\dfrac{\Delta f(x)}{f(x)}\right\}$. 3

(d) Find the Newton Raphson formula to find $\sqrt[3]{N}$, where N is a positing integer. 3

(e) What is Regular-Falsi method for numerical solution of the equation $f(x)=0$. Interprete this geometrically. 3

(f) How should the constant α be chosen to ensure the fastest possible convergence with the iteration formula 3

$$x_{n+1} = \frac{\alpha x_n + x_n^{-2} + 1}{\alpha + 1}\ ?$$

GROUP-B

2. Answer any ***four*** questions from the following: $6\times 4 = 24$

(a) Find positive roots of the equation $x^3 - 9x + 1 = 0$ lying between 2 and 3 upto two significant digits by method of bisection. 6

(b) Evaluate $\int_0^1 (5x-3x^2)dx$, taking 10 intervals by using trapezoidal Rule. Also compute the exact value and, find the absolute and relative errors. 6

(c) Explain the method of bisection for computing a real root of an equation $f(x)=0$ with its advantages and disadvantages. 6

(d) Find the roots of $2x-\log_{10} x-7=0$ by the method of iteration correct to four decimal places. 6

(e) Use Picard's method to compute $y(0.1)$ and $y(0.2)$ from the equation $\frac{dy}{dx}=1+xy$, $y(0)=1$. 6

(f) Solve by Gauss-Seidel iteration method, the system: 6

$$3x_1+9x_2-2x_3=11$$
$$4x_1+2x_2+13x_3=24$$
$$4x_1-2x_2+x_3=-8$$

correct upto four significant figures.

GROUP-C

Answer any *two* questions from the following 12×2 = 24

3. (a) Solve by using suitable interpolation formula to find $f(10)$ correct upto three places of decimal from the following value of $f(x)$: 6

x	5	6	9	11
$f(x)$	12	13	14	16

(b) If a number is corrected to n significant figures and the first significant figure of the number is k, then show that the relative error 6

$$\varepsilon_r < \frac{1}{k\cdot 10^{n-1}}.$$

4. (a) Solve by Euler's modified method the following equation for $x=0.02$ by taking step length $h=0.01$: $\frac{dy}{dx}=x^2+y$, $y(0)=1$. 6

(b) Evaluate $\int_0^{\pi/2} \sqrt{\cos x}\, dx$, by Simpson's one-third rule taking $n=6$. 6

5. (a) Find the real roots of the equation $x^3+x^2-1=0$ using fixed point iteration method, correct upto six decimal places. 6

(b) Explain Newton-Raphson method and its convergence. 6

6. (a) Calculate the value of y when $x=1.6$ from the following data table: 6

x	0	4	8	12	16
y	1.0	1.5	3.6	7.9	14.6

(b) Solve the system of linear equations by using Gauss-Jordan method: 6

$$3x_1+2x_2+3x_3=18$$
$$2x_1+x_2+x_3=10$$
$$x_1+4x_2+9x_3=16$$

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