'समानो मन्त्रः समितिः समानी'

# UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2022

## CC5-MATHEMATICS

### THEORY OF REAL FUNCTIONS AND INTRODUCTION TO METRIC SPACE

Time Allotted: 2 Hours | Full Marks: 60

*The figures in the margin indicate full marks.*

### GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) A function $f:[0,1]\to[0,1]$ is continuous on $[0,1]$. Prove that there exists a point $c$ in $[0,1]$ such that $f(c)=c$. 3

(b) Show that $\lim\limits_{x\to 0}\dfrac{x-|x|}{2}$ does not exist. 3

(c) Prove that if $f(x)$ is continuous at $x=a$ and for every $\delta>0$ there is a point in $|x-a|<\delta$, where $f(x)=0$ then $f(a)=0$. 3

(d) Verify Rolle's theorem for $f(x)=2x^3+x^2-4x-2$. 3

(e) Prove that for two subsets $A, B$ of a metric space $(X, d)$ if $A\subseteq B$, then $\delta(A)\le\delta(B)$. 3

(f) In a metric space $(X, d)$ if $a, b\in X$ and $a\neq b$, then show that there exists open balls $S_a$ and $S_b$ containing *a* and *b* respectively such that $S_a\cap S_b=\varphi$. 3

### GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) If $g(x)=f(x)+f(1-x)$ and $f''(x)<0$ on $[0,1]$, show that $g(x)$ is monotonically increasing on $\left[0,\frac{1}{2}\right]$ and monotonically decreasing on $\left[\frac{1}{2},1\right]$. 6

(b) Show that $\dfrac{v-u}{1+v^2}<\tan^{-1}v-\tan^{-1}u<\dfrac{v-u}{1+u^2}$ if $0<u<v$ and deduce that $\dfrac{\pi}{4}+\dfrac{3}{25}<\tan^{-1}\left(\dfrac{4}{3}\right)<\dfrac{\pi}{4}+\dfrac{1}{6}$ 4+2

(c) Let $\lim\limits_{x\to a}\varphi(x)=l$ and $f$ is continuous at *l*. Prove that 6

$$\lim_{x\to a}f(\varphi(x))=f(\lim_{x\to a}\varphi(x))=f(l).$$

(d) State and prove Darboux's theorem. 6

(e) Let $(X, d)$ be a complete metric space and $Y$ be a subspace of $X$. Prove that $Y$ is complete if and only if $Y$ is closed. 6

(f) Let $X = \ell_p\ (1 \le p < \infty) =$ The set of all $p$th summable sequences of real or complex numbers and let $d(x, y) = \left\{\sum_{i=1}^{\infty} |x_i - y_i|^p\right\}^{1/p}$ where $x = \{x_n\}$ and $y = \{y_n\} \in \ell_p$. Prove that '$d$' is a metric on $X = \ell_p$. 6

## GROUP-C

3. Answer any ***two*** questions from the following: 12×2 = 24

(a) (i) Find the power series expansion of $\log(1+x)$. Stating clearly its region of validity. 6

(ii) Let $(Y, d')$ be a subspace of a metric space $(X, d)$. Prove that a set $A \subset Y$ is open in $(Y, d')$ if and only if there exists an open set $G$ in $(X, d)$ such that $A = G \bigcap Y$. 6

(b) (i) Define separable metric space. Give an example of separable metric space with justification. 1+5

(ii) State and prove Cantor's intersection theorem. 1+5

(c) (i) Show that the function $f$ where $f(x)\begin{cases} x\,[1+\frac{1}{3}\sin(\log x^2)], & x \ne 0 \\ 0, & x = 0 \end{cases}$ 6

is continuous everywhere and monotonic but has no differential coefficient at $x = 0$.

(ii) Let $(X, d)$ be a metric space and $A \subset X$. Show that $\alpha \in \overline{A}$ if and only if $S \bigcap A \ne \phi$ for every neighbourhood $S$ of $\alpha$. 6

(d) (i) Use mean value theorem of appropriate order to prove that 6

$x > \sin x > x - \frac{1}{6}x^3,\ 0 < x < \pi/2$.

(ii) Let $(X, d)$ be a metric space. Prove that a non-empty set $A \subset X$ is nowhere dense in $X$ if and only if the set $(\overline{A})^c = X \setminus \overline{A}$ is dense in $(X, d)$. 6

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