'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2022

CC6-MATHEMATICS

GROUP THEORY-I

Time Allotted: 2 Hours Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) Find the number of elements of order 5 in $\mathbb{Z}_{15}\times\mathbb{Z}_5$.

(b) Let $\beta=(1\ 2\ 3)(1\ 4\ 5)$, write β^{99} in cycle notation.

(c) Prove that a group G in which $a^2=e$ for every element a in G is a commutative group, where e is an identity element of G.

(d) Find all homomorphism from the group $(\mathbb{Z}_6, +)$ to $(\mathbb{Z}_4, +)$.

(e) Prove that centre of the symmetric group S_3 is trivial.

(f) Prove that a non-abelian group of order 8 must have an element of order 4.

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) Prove that the order of every subgroup of a finite group G is a divisor of the order of G. 4+2

Is the converse true?

(b) (i) Prove that up to isomorphism, there are only two groups of order 4. 4

(ii) Let $G\neq\{e\}$ be a group of order p^n, p is prime. Show that G contains an element of order p. 2

(c) Let H be a subgroup of a group G and $[G:H]=2$. Prove that for every $x\in G$, $x^2\in H$. 4+2

Deduce that A_4 has no subgroup of order 6.

(d) (i) Show that a group G of even order contains an odd number of elements of order 2. 4

(ii) Let in a group G, a be an element of order 30. Find $o(a^{18})$. 2

(e) In the direct product $\mathbb{Z}_{20}\times \mathbb{Z}_2\times \mathbb{Z}_8\times \mathbb{Z}_6$, how many elements of order 12 exists? 6

(f) (i) Let G be a finite commutative group of order n and $\gcd(m,n)=1$. Prove that $\phi: G\to G$ defined by $\phi(x)=x^m$, $x\in G$ is an isomorphism. 3

(ii) Prove that the external direct product of two groups A and B is commutative if and only if both groups A and B are commutative. 3

GROUP-C

3. Answer any ***two*** questions: 12×2 = 24

(a) (i) Let H be a subgroup of a group G. Then prove that $K=\bigcap_{g\in G} gHg^{-1}$ is a normal subgroup of G. 4

(ii) Let $\alpha=\begin{pmatrix}1&2&3&4&5&6&7\\6&4&7&5&2&3&1\end{pmatrix}$, $\beta=\begin{pmatrix}1&2&3&4&5&6&7\\1&4&6&7&3&5&2\end{pmatrix}$ be the elements of S_7. 3+3+2

(A) Write α as a product of disjoint cycles.

(B) Write β as a product of 2-cycles.

(C) Is α^{-1} an even permutation?

(b) (i) Let G and G' be two groups and $\phi: G\to G'$ be onto homomorphism. If $H=\ker\phi$, then prove that $G/H\simeq G'$. 6

(ii) Show that a finite semigroup in which cancellation laws hold is a group. 6

(c) (i) If N and M are normal subgroups of G, then prove that 3+3

(A) $N\cap M$ is also normal in G.

(B) NM is also normal in G.

(ii) Prove that $\mathbb{Z}/3\mathbb{Z}\simeq \mathbb{Z}_3$. 3

(iii) Find the order of H if H is a proper subgroup of a group of order 68 and H is non-cyclic. 3

(d) (i) Prove that the group $S_n (n\geq 3)$ is not abelian. 6

(ii) Show that a group homomorphism $\psi:(G,\circ)\to(H,*)$ is one-one if and only if $\ker\psi=\{e\}$. Deduce that the homomorphism $\phi:(\mathbb{Z}_6, +)\longrightarrow(\mathbb{Z}_6, +)$ defined by $\phi(\bar{x})=\overline{2x}$ is not one-one. 4+2

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