'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2022

CC7-MATHEMATICS

RIEMANN INTEGRATION AND SERIES OF FUNCTIONS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) Consider the function: $f(x)=\begin{cases}\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$. 3

Show that the improper integral $\int_{-1}^{1} f(x)dx$ does not exist whereas its Cauchy principle value exists.

(b) Show that $\Gamma(x)>\frac{1}{e}\int_{0}^{1} t^{x-1}dt=\frac{1}{e^{x}}$ for $x>0$. 3

(c) Examine the uniform convergence of the series $\sum\frac{(-1)^{n}}{n}|x|^{n}$ in $-1\leq x\leq 1$. 3

(d) Find the radius of convergence and exact interval of convergence of the power series $\sum\frac{(n+1)}{(n+2)(n+3)}x^{n}$. 3

(e) Express $f(x)=\frac{\pi-x}{2}$ in a Fourier series in $0<x<2\pi$. 3

(f) Show that the function 3

$$f(x)=\begin{cases}0, & \text{when } x \text{ is rational}\\ 1, & \text{when } x \text{ is irrational}\end{cases}$$

is not integrable on any interval.

GROUP-B

Answer any *four* questions 6×4 = 24

2. Test the convergence of $\int_{0}^{\infty}\frac{x\,dx}{1+x^{4}\cos^{2}x}$. 6

3. Find the Fourier series for the function $f(x)=\left|\cos\left(\frac{\pi x}{l}\right)\right|$ of period $2l>0$. 6

4. (a) Consider the function f defined on $\left[0,\frac{\pi}{2}\right]$ as follows: 3

$$f(x)=\begin{cases}\cos^2 x, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

Examine the Riemann integrability of f on $\left[0,\frac{\pi}{2}\right]$.

(b) Give an example of a function which is integrable but has no anti-derivative. 3

5. Show that the sequence of functions $\{f_n\}$, where $f_n(x)=x^n$, is uniformly convergent in $[0,k]$, $k<1$ and pointwise convergent in $[0,1]$. 6

6. Discuss the convergence of $\int_0^{\pi/2}\log(\sin x)dx$. Hence find its value, if possible. 3+3

7. Let $\sum_{n=0}^{\infty}a_n x^4$ be a power series with radius of convergence $R(>0)$ and $f(x)$ be the sum of the series on $(-R,R)$, show that $f^k(0)=k!a_k$ $(k=0,1,2,\cdots)$. 6

GROUP-C

Answer any *two* questions 12×2 = 24

8. (a) Show that $\int_0^{\pi/2}\frac{x^m}{\sin^n x}dx$ is convergent iff $n<1+m$. 4

(b) A sequence of functions $\{f_n\}$ is defined by $f_1(x)=\sqrt{x}$, $f_{n+1}(x)=\sqrt{x f_n(x)}$ $\forall\, n\geq 1$. 4

Show that $\{f_n\}$ is uniformly convergent on $[0,1]$.

(c) Prove that two different power series cannot converge on the same interval $(-R,R), R>0$ to the same function f. 4

9. (a) Obtain the Fourier series expansion of $f(x)=x\sin x$ on $[-\pi,\pi]$. Hence deduce that $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots$. 7

(b) Assuming $\sin^{-1}x=x+\frac{1}{2}\,\frac{x^3}{3}+\frac{1.3}{2.4}\,\frac{x^5}{5}+\cdots$ for $-1\leq x\leq 1$, prove that 5

$$\int_0^1\frac{\sin^{-1}x}{x}dx=1+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1.3}{2.4}\,\frac{1}{5^2}+\cdots$$

10.(a) State and prove Darboux's Theorem. 1+5

(b) If f is bounded and integrable in $[-\pi, \pi]$ and monotonic in $[-\delta, 0)$ and $(0, \delta]$, where $0 < \delta < \pi$, then show that $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{f(0^-) + f(0^+)}{\pi} \int_0^{\infty} \frac{\sin x}{x} dx$ where $a_n (n = 0, 1, 2, \cdots)$ denote the Fourier coefficients of f. 6

11.(a) Test the convergence of $\int_0^{\infty} e^{-x} x^{n-1} dx$. 7

(b) Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n(1+x^n)}$ is uniformly convergent on $[0, 1]$. 5

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