'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2022

GE2-P1-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

The question paper contains MATHGE1, MATHGE2, MATHGE3, MATHGE4 and MATHGE5. Candidates are required to answer any *one* from the *five* MATHGE courses and they should mention it clearly on the Answer Book.

MATHGE1

CALCULUS, GEOMETRY AND DE

GROUP-A

1. Answer any ***four*** questions from the following: $3\times4 = 12$

(a) If $y=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$, find y_n.

(b) Find the area bounded by the curve

$$y=x(x-1)(x-2) \text{ and the } x\text{-axis.}$$

(c) Determine the type of conic represented by the equation

$$5x^2-6xy+5y^2+22x-26y+29=0$$

(d) If $y=x^{n-1}\ \log x$, then prove that $y_n=\dfrac{(n-1)!}{x}$

(e) Find the limit $\lim\limits_{x\to0}\dfrac{\tan x-x}{x-\sin x}$.

(f) Obtain the differential equation of all circles each of which touches the axis of x at the origin.

GROUP-B

2. Answer any ***four*** questions from the following: $6\times4 = 24$

(a) Find the reduction formula for 4+2

$$\int\sec^n x\,dx,\ n\in\mathbb{N},\ n>1.$$

Hence obtain $\int\sec^4 x\,dx$.

(b) Find the area of the entire surface formed when the cardioid $r = a(1+\cos\theta)$ is revolved about the initial line. 6

(c) Find the equation of the sphere passing through the point $(1, 0, -3)$ and through the circle represented by 6

$$x^2 + y^2 + z^2 - 4x - 6y + 2z = 16,\ 3x + y + 3z - 4 = 0.$$

(d) Find the asymptotes of the curve $(x^2 - y^2) - 8(x^2 + y^2) + 8x - 16 = 0$ 6

(e) Find the envelope of the circles $x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$, where α, β are parameters and whose centres lie on the parabola $y^2 = 4ax$. 6

(f) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$, when $y = \dfrac{1}{3}$ at $x = \dfrac{2}{3}$. 6

GROUP-C

3. Answer any ***two*** questions from the following: 12×2 =24

(a) (i) If $I_n = \int_0^{\pi/2} x \sin^n x\, dx$, $n > 1$, $n \in \mathbb{N}$, then prove that $I_n = \dfrac{n-1}{n} I_{n-2} + \dfrac{1}{n^2}$ 6+6

(ii) Prove that $\int_0^{\pi/2} \sin 2nx\, dx = \pi/2$

(b) (i) Find the volume of the solid obtained by revolving $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis 6+6

(ii) PSP' is a focal chord of the conic $\dfrac{l}{r} = 1 - e\cos\theta$, Prove that the angle between the tangents at P and P' is $\tan^{-1}\left(\dfrac{2e\sin\alpha}{1-e^2}\right)$

(c) (i) If by a rotation of coordinate axes the expression $ax + by$, $cx + dy$ changed to $a'x' + b'y'$, $c'x' + d'y'$ respectively, then show that $a'd' - b'c' = ad - bc$ 6+6

(ii) Reduce the equation $3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$ to the canonical form.

(d) (i) Solve: $x^3 \dfrac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$ 6+6

(ii) Solve: $x\cos x \dfrac{dy}{dx} + y(x\sin x + \cos x) = 1$.

MATHGE2

ALGEBRA

GROUP-A

1. Answer any ***four*** questions: $3\times4=12$

(a) Find the minimum value of $(3x+2y)$ where x, y are positive real numbers satisfying the condition $x^2y^3=48$. 3

(b) Use De Moivre's theorem to find the roots of the equation $x^5-1=0$ 3

(c) If $A=\begin{pmatrix}0 & 4 & 0\\ 0 & 0 & 1\\ 4 & 0 & 0\end{pmatrix}$, find the rank of the matrix $A+A^2$ 3

(d) Solve: $58x\equiv 6(\text{mod}\,78)$ 3

(e) Use Division Algorithm, prove that the square of an odd integer is of the form $(8k+1)$, where k is an integer. 3

(f) Examine whether T is a linear transformation, where $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ defined by $T(x, y, z)=(x+2y+3z, 3x+2y+z, x+y+z)$, $(x, y, z)\in\mathbb{R}^3$. 3

GROUP-B

2. Answer any ***four*** questions: $6\times4=24$

(a) If $x=\log\tan\left(\pi/4+\frac{1}{2}y\right)$, then prove that $y=-i\log\tan\left(\frac{\pi}{4}+\frac{1}{2}ix\right)$ 6

(b) Find the conditions that the roots of the equation $x^3-px^2+qx-r=0$ will be in (i) A.P. (ii) G.P. 6

(c) Prove that $\frac{1}{2\sqrt{n+1}}<\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}<\frac{1}{\sqrt{2n+1}}$ 6

(d) If λ be an eigenvalue of a real orthogonal matrix A, prove that $\frac{1}{\lambda}$ is also an eigenvalue of A. 6

(e) Prove that the linear mapping $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ that maps the basis vectors $(1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 1)$ of $\mathbb{R}^3$ to the vectors $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ respectively is one-one and onto. 6

(f) Let $f:A\rightarrow B$ and $g:B\rightarrow C$ be two mappings. Prove that if $g\circ f$ is surjective and g is injective then f is surjective. 6

GROUP-C

3. Answer any ***two*** questions: 12×2= 24

(a) (i) Examine if the relation ρ on the set $\mathbb{Z}$ is (i) reflexive (ii) Symmetric (iii) transitive, where ρ is defined as $a\rho b$ if and only if $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ and $(a^2 + b^2)$ is a multiple of 2. 2+2+2

(ii) Use the principle of induction to prove that 6

$1.1! + 2.2! + \cdots + n.n! = (n+1)! - 1$, for all $n \in \mathbb{N}$

(b) (i) Find the eigen values and the eigen vectors of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{pmatrix}$ 6

(ii) Solve by Cardan's method the equation: $x^3 - 18x - 35 = 0$ 6

(c) (i) If x, y, z be any three real numbers then show that 6

$$\frac{x^2 + y^2}{x+y} + \frac{y^2 + z^2}{y+z} + \frac{z^2 + x^2}{z+x} \geq x + y + z$$

(ii) A linear mapping $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by 6

$T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, 2x_1 + 3x_2 + x_3, 3x_1 + x_2 + 2x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Find the matrix of T relative to the ordered basis

$\{(-1,1,1), (1,-1,1), (1,-1,1)\}$ of $\mathbb{R}^3$.

(d) (i) Determine the value of a and b for which the system of equations 6

$$x + y + z = 1$$

$$x + 2y - z = b$$

$$5x + 7y + az = b^2$$

has (I) no solution (II) unique solution (III) many solution.

(ii) Apply Descarte's Rule of signs to show that the equation $x^4 + 2x^2 - 7x - 5 = 0$ has only two real roots. Find these two real roots. 6

MATHGE3

DE AND VECTOR CALCULUS

GROUP-A

1. Answer any ***four*** questions from the following: $3\times4 = 12$

(a) Show that $f(x)=\log x$, $x\in(0,\infty)$ is a Lipschitz function on $[a,\infty)$, where $a>0$. 3

(b) Evaluate $\frac{1}{D+1}(x^2+1)$, where $D\equiv\frac{d}{dx}$. 3

(c) Examine whether $x=0$ is a regular singular point of the differential equation : $y''-xy'+2y=0$. 3

(d) Evaluate: $\lim_{t\to\infty}\left(\frac{t^3+1}{4t^3+2}\hat{i}+\frac{1}{t}\hat{j}\right)$ 3

(e) Examine whether the vector valued function $\vec{r}=t^3\hat{i}+e^t\hat{j}+\frac{1}{t+3}\hat{k}$ is continuous or not at $t=3$ 3

(f) Suppose $A=5t^3\hat{i}+t^2\hat{j}-t\hat{k}$ and $B=\sin t\hat{i}-\cos t\hat{j}$. Find $\frac{d}{dt}(A\times B)$. 3

GROUP-B

2. Answer any ***four*** questions from the following: $6\times4 = 24$

(a) Solve: $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-y=3x^3\cos(\log x)$ 6

(b) Solve: $(D+2)(D-1)^3y=e^x$ 6

(c) Find the Wronskian of $\left\{x^3+2,\ \frac{|x|}{5}\right\}$. Show whether these functions are linearly independent or not. 6

(d) Find a vector function $\vec{F}$ whose graph is the curve $\frac{x-2}{3}=\frac{y-1}{2}=\frac{z}{4}$. 6

(e) Find the coordinates of the point where the line $\vec{r}=t\hat{i}+(1+2t)\hat{j}-3t\hat{k}$ intersects the plane $3x-y-z=2$. 6

(f) Prove that

$$[\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} \vec{l}-\vec{a} & \vec{l}-\vec{b} & \vec{l}-\vec{c} \\ \vec{m}-\vec{a} & \vec{m}-\vec{b} & \vec{m}-\vec{c} \\ \vec{n}-\vec{a} & \vec{n}-\vec{b} & \vec{n}-\vec{c} \end{vmatrix}$$ and hence deduce that

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}-\vec{a} & \vec{a}-\vec{b} & \vec{a}-\vec{c} \\ \vec{b}-\vec{a} & \vec{b}-\vec{b} & \vec{b}-\vec{c} \\ \vec{c}-\vec{a} & \vec{c}-\vec{b} & \vec{c}-\vec{c} \end{vmatrix}.$$

GROUP-C

3. Answer any ***two*** questions from the following: 12×2= 24

(a) (i) Solve the system of equations: 6+6

$$\frac{dy}{dx} + 2y - 3z = x$$

$$\frac{dz}{dx} + 2z - 3y = e^{2x}$$

(ii) Solve the following system of linear equations by using operator $D \equiv \frac{d}{dx}$

$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0$$

$$\frac{dy}{dt} + 5x + 3y = 0$$

(b) (i) Solve by method of variation of parameters: 6+6

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x, \ (x > 0).$$

(ii) Solve by using the method of undetermined coefficient:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 10e^{2x} - 18e^{3x} - 6x - 11$$

(c) (i) Show that the vector function $\vec{r}(t) = \begin{cases} \frac{\sin t}{t}\hat{i} + \cos t\,\hat{j} + t^3\hat{k}, & t \neq 0 \\ \hat{i}, & t = 0 \end{cases}$ 4+6+2

is continuous at $t = 0$.

(ii) Evaluate $\int_1^2 \vec{a}\cdot(\vec{b}\times\vec{c})dt$,

where $\vec{a} = t\hat{i} - 3\hat{j} + 2t\hat{k}$

$\vec{b} = \hat{i} - 2\hat{j} + 2\hat{k}$

$\vec{c} = 3\hat{i} + t\hat{j} - \hat{k}$

(iii) Define continuity of a vector valued function.

(d) (i) If $\vec{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, 6+6

Prove that $\int_1^2 \vec{r}\times\frac{d^2\vec{r}}{dt^2}dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$

(ii) Prove that the necessary and sufficient condition for $\vec{f}(t)$ to have constant direction is $\vec{f}\times\frac{d\vec{f}}{dt} = \vec{O}$.

MATHGE4

GROUP THEORY

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) If $G = S_3$ and $H = \{I, (1, 3)\}$, (I being the identity permutation). Write all left cosets of H in G. 3

(b) Give example of two subgroups H, K which are not normal, but HK is a subgroup. 3

(c) Show that in a group $(G, \circ)$, $\forall a, b \in G$, each of the equations $a \circ x = b$ and $y \circ a = b$ has a unique solution in G. 3

(d) Prove that, if $a^2 = e$ for all elements $a \in G$, where G is a group, then G is commutative. 3

(e) Let $\mathbb{Q}^* = \mathbb{Q} - \{0\}$, where $\mathbb{Q}$ is the set of rational numbers. Show that $(\mathbb{Q}, +)$ cannot be isomorphic to $(\mathbb{Q}^*, \cdot)$. 3

(f) If in a group G, $a^5 = e$, $aba^{-1} = b^2$ for $a, b \in G$, then show that $o(b) = 31$. 3

GROUP-B

Answer any *four* questions 6×4 = 24

2. Consider the Dihedral group $D_4 = \{a, b : a^4 = b^2 = e, ab = ba^{-1}\}$. Find the centre of D_4 and $C(a)$. 6

3. If H is a subgroup of a commutative group G, then prove that the quotient group G/H is commutative. Is the converse true? Justify. 6

4. (a) Find the order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$. Determine whether this permutation is even or odd. 3

   (b) Correct or justify that every group of order 6 is commutative. 3

5. Prove that the order of a subgroup of a finite group divides the order of the group. 6

6. (a) Show that A_4 is the only subgroup of order 12 in S_4. 3

   (b) Find the order of each element in the group $G = \{\pm 1, \pm i\}$ under multiplication. 3

7. Let G be a group such that $\forall\, a, b \in G$, $(a\,b)^3 = a^3 b^3$ and $(ab)^5 = a^5 b^5$. Show that G is abelian. 6

GROUP-C

Answer any *two* questions 12×2= 24

8. (a) Prove that the set $G = \{2^n : n \in \mathbb{Z}\}$ forms a group with respect to multiplication. 4

   (b) Find all elements of order 8 in the group $(\mathbb{Z}_{24}, +)$. 3

   (c) Define a binary compositions '$\circ$' on $\mathbb{Z}$ by $a \circ b = a + b - ab$ $\forall\, a, b \in \mathbb{Z}$. Show that $(\mathbb{Z}, \circ)$ is a monoid. 3

   (d) Examine whether $M_2(\mathbb{R})$ (the set of all 2×2 matrices whose elements are real no's) forms a group or not under matrix multiplication. 2

9. (a) A finite group $(G, \circ)$ of order n is cyclic iff there exists an element $b \in G$ such that $o(b) = n$. Prove it. 6

   (b) Write down the set of invertible elements U_{10} from $(\mathbb{Z}_{10} \setminus \{\overline{0}\}, \cdot)$ and show that $(U_{10}, \cdot)$ is a cyclic group. 6

10. (a) Let $\phi : (G, \circ) \to (G', *)$ be a homomorphism. Then prove that $\ker\phi$ is a normal subgroup of G. 4

(b) Let $\phi:(G,\circ)\to(G',*)$ be a homomorphism. Then show that ϕ is one-one iff $\ker\phi=\{e_G\}$. 4

(c) Let $G=(\mathbb{R},\ +)$ and $G'=(\{z\in\mathbb{C}\ :|z|=1\},\ \cdot)$ and $\phi:G\to G'$ is defined by $\phi(x)=\cos(2\pi x)+i\sin(2\pi x)\ \ \forall x\in\mathbb{R}$. Prove that ϕ is a homomorphism. 4

11.(a) If H is a normal subgroup of a group G and L be a subgroup of G such that $H\subseteq L\subseteq G$. Prove that H is normal in L. 6

(b) Let H be a subgroup of G and $[G:H]=2$. Prove that 4+2

(i) $\forall\, x\in G,\ x^2\in H$

(ii) H is a normal subgroup of G.

MATHGE5

NUMERICAL METHODS

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) If $U=\dfrac{5xy^2}{z^3}$ and errors in x,y,z are 0.001, compute the relative maximum error in U when $x=y=z=1$

(b) Show that n^{th} order forward difference of x^n is constant.

(c) Prove that $\Delta\nabla=\Delta-\nabla$, where the symbols are in usual meaning.

(d) Write the sufficient condition for convergence of Gauss-Seidel iteration method.

(e) What is the geometrical interpolation of Simpson's one-third rule?

(f) Find $y(0.02)$, by Euler's method from the differential equations $\dfrac{dy}{dx}=x^3+y$, when $y(0)=1$, correct upto four decimal places, taking step length $h=0.01$.

GROUP-B

Answer any *four* questions 6×4 = 24

2. If a number is connected to n significant figures and the first significant figure of the number is k, then prove that the relative error $\varepsilon_r<\dfrac{1}{k\cdot 10^{n-1}}$

3. Compute $y(0.8)$, by Runge-Kutta method correct up to five decimal places, from the equation

$$\frac{dy}{dx}=xy,\ y(0)=2\ \text{taking}\ h=0.2$$

4. Explain the method of fixed point iteration with the condition of convergence for numerical solution of an equation of the form $x = \phi(x)$.

5. Use Gauss-Jacobi method to solve

$$5x - y + z = 10$$
$$2x + 4y = 12$$
$$x + y + 5z = -1$$

6. Compute $\int_0^1 (4x - 3x^2)dx$ upto five decimal places by taking 10 intervals by trapezoidal rule.

7. Prove that $\Delta^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!\, h^n}{x(x+h)\cdots(x+nh)}$ for any positive integer n, Δ being forward difference operator and h the step length.

GROUP-C

Answer any *two* questions 12×2=24

8. (a) Evaluate $\int_1^4 \frac{\log_e(1+0.5x+x^2)}{0.5+x}dx$ by Simpson's $\frac{1}{3}$rd rule, correct upto 6 decimal places, taking 11 ordinates. 6

(b) Using Newton-Raphson method find a positive root of the equation $e^x - 3x = 0$ correct upto four decimal places. 6

9. (a) Find $y(4.4)$ by Euler's modified method, taking $h = 0.2$ from the differential equation: 6

$$5x\frac{dy}{dx} + y^2 = 2,\ y(4) = 1.$$

(b) Solve by Gauss-Seidel iteration method, the system 6

$$3x_1 + 9x_2 - 2x_3 = 11$$
$$4x_1 + 2x_2 + 13x_3 = 24$$
$$4x_1 - 2x_2 + x_3 = -8$$

correct upto four significant figures.

10.(a) Evaluate the missing terms in the following table: 6

x	2	3	4	5	6	7	8
$f(x)$	0.135	?	0.111	0.100	?	0.082	0.074

(b) What is interpolation? Establish Newton's forward interpolation formula. 6

11.(a) Define backward difference operator ∇ and shifting operator E. Show that 6

$$\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0 .$$

(b) Solve the system of equation by Gauss-elimination method. 6

$$2x_1 + 3x_2 + x_3 = 9$$

$$x_1 + 2x_2 + 3x_3 = 6$$

$$3x_1 + x_2 + 2x_3 = 8$$

correct upto 3-significant figures.

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