'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 5th Semester Examination, 2022

CC11-MATHEMATICS

GROUP THEORY-II

Time Allotted: 2 Hours — Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Give an example of a group G_1 and G_2 such that $G_1 \ncong G_2$, but $\text{Aut}(G_1) \cong \text{Aut}(G_2)$. 3

(b) Prove that if G is a finite group, then G is a p-group if and only if $o(G) = p^n$. 3

(c) Let $o(G) = 30$. Show that either sylow-3-subgroup or sylow-5-subgroup is normal in G. 3

(d) Show that $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$. 3

(e) Show that if G is non-abelian group then the mapping $f: G \to G$ defined by $f(x) = x^{-1}$ is not an automorphism. 3

(f) Let G be a finite group that has only two conjugate classes. Show that order of the group G is 2. 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) (i) If $Z(G)$ be the centre of a group G, then prove that $G/Z(G) \cong \text{Inn}(G)$. 3+3

(ii) Find all abelian groups (up to isomorphism) of order 360.

(b) Show that for any prime p, there exists only two non-isomorphic groups of order p^2. 6

(c) State and prove Cauchy's theorem. 6

(d) Let G be a group. Let $\text{Aut}(G)$ denotes the set of all automorphisms of G and $A(G)$ be the set of all permutations on G. Then prove that $\text{Aut}(G)$ is a subgroup of $A(G)$. 6

(e) Let G be any group and A be a non-empty set. Then prove that any homomorphism from G to Sym(A), (where Sym(A) is symmetric group of A) defines an action of G on A. Conversely every action of G on A induces a homomorphism from G to Sym(A). 6

(f) Let G be a simple group of order 168. Find the number of subgroups of order 7. 6

GROUP-C

3. Answer any ***two*** questions from the following: 12×2 = 24

(a) (i) Let G be a group of order p^2, where p is a prime. Then show that G is commutative. 4+4+4

(ii) Find $\text{Aut}(K_4)$.

(iii) Let G be a group and S be a G-set. Show that for all $a \in S$, the subset $G_a = \{g \in G : ga = a\}$ is a subgroup of G.

(b) (i) Prove that there are only two non-commutative groups of order 8 (up to isomorphism). 5+3+4

(ii) Let S be a finite G-set, where G is a group of order p^n (p is a prime). Show that $|S| \equiv_p |S_0|$, where $S_0 = \{a \in S \mid ga = a \ \forall \ g \in G\}$.

(iii) Let G be a group and $a \in G$. Prove that $a \in Z(G)$ if and only if $\text{Cl}(a) = \{a\}$.

(c) (i) Let K be a subgroup of H and H be a subgroup of G. Suppose K is a characteristic subgroup of G and H/K is a characteristic subgroup of G/K. Then show that H is a characteristic subgroup of G. 6+6

(ii) Let G be a group of order 231. Show that 11-sylow subgroup of G is contained in the centre of G.

(d) (i) Let G be a group of order pqr, where $p < q < r$ being primes. Prove that some sylow subgroup of G is normal. Hence show that G is not simple. 6+6

(ii) Let G be a group that acts on a set A. Let $a \in A$, then prove that $|G| = |\text{Stab}(a)| \times |\text{Orbit}(a)|$.

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