'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 5th Semester Examination, 2022

DSE-P2-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.

The question paper contains DSE2A and DSE2B. Candidates are required to answer any *one* from the *two* DSE2 courses and they should mention it clearly on the Answer Book.

DSE2A

NUMBER THEORY

GROUP-A

1. Answer any ***four*** questions: $3\times4 = 12$

(a) Find the inverse of 7 (mod 20). 3

(b) Find the highest power of 11 dividing 1000. 3

(c) Find the integer in the unit place of 2^{15}. 3

(d) Verify Division Algorithm for Gaussian integers using $12+8i$ as dividend and $4-i$ as divisor. 3

(e) Prove that for any integer a, $a^3 = 0, 1$ or 6 (mod 7). 3

(f) Define Legendre Symbol. 3

GROUP-B

2. Answer any ***four*** questions: $6\times4 = 24$

(a) If $x \equiv a \pmod{16}$, $x \equiv b \pmod{5}$ and $x \equiv c \pmod{11}$, prove that 6

$$x \equiv 385a + 176b - 560c \pmod{880}$$

(b) Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integral coefficients such that $a_n \not\equiv 0 \pmod{p}$, p being a prime. Prove that the congruence $f(x) \equiv 0 \pmod{p}$ has at most n incongruent solutions. 6

(c) If $2^n - 1$ is prime, prove that n must be a prime. Is the converse true? Justify. 6

(d) Solve: $x^2 + 7x + 10 \equiv 0 \pmod{11}$ 6

(e) Find $\gcd(-3+11i, 8-i)$ in $\mathbb{Z}[i]$. Also, find $x, y \in \mathbb{Z}[i]$ such that 6

$$\gcd(-3+11i, 8-i) = (-3+11i)x + (8-i)y \ .$$

(f) State and prove Fermat's two square theorem. 6

GROUP-C

3. Answer any ***two*** questions: 12×2 = 24

(a) (i) For any two integers a and n with $\gcd(a,n)=1$, prove that $a^{\phi(n)} \equiv 1 \pmod{n}$. 6

(ii) If p and q are distinct primes, then prove that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$ 6

(b) (i) Prove that n is a prime iff $(n-1)! \equiv -1 \pmod{p}$ 8

(ii) Define a quadratic residue. State Quadratic Reciprocity Law. 2+2

(c) (i) Find the number of positive integers less than 700 that are divisible by atleast one of the primes 3, 5 and 7. 6

(ii) Prove that $1+2+3+\cdots+n$ is a divisor of $1^r + 2^r + \cdots + n^r$ for any positive integer r. 6

(d) (i) Consider the linear Diophantine equation $ax+by=c$, where $a, b, c \in \mathbb{Z}$. Show that the equation admits a solution iff $\gcd(a,b) \mid c$. 7

If (x_0, y_0) is a particular solution of the equation, find the general solution.

(ii) Solve : $x^2 \equiv 14 \pmod{5^3}$. 5

DSE2B

MECHANICS

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) Define stable and unstable equilibrium. Write down statement of energy test of stability.

(b) Show that the momental ellipsoid at the centre of an ellipsoid is

$$(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b^2)z^2 = \text{Constant.}$$

(c) Prove that the squares of the periodic times of the planets are proportional to the cubes of the mean distance from the Sun.

(d) A uniform cubical box of edge 'a' is placed on the top of a fix sphere. Show that the least radius of the sphere for which the equilibrium will be stable is $\frac{a}{2}$.

(e) State principle of virtual work.

(f) Define equimomental system. Under what conditions two systems will be equimomental?

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) A particle is projected with velocity u at an inclination α above the horizon in a medium whose resistance per unit mass is k times the velocity. Show that its direction will again make an angle α below the horizon after a time. 6

$$\frac{1}{k}\log\left\{1+\frac{2ku}{g}\sin\alpha\right\}$$

(b) Define compound pendulum. Show that the centre of suspension and oscillation of a compound pendulum are interchangeable. 1+5

(c) A body rests in equilibrium on another fixed body being enough friction to prevent sliding. The portion of the two bodies in contact are spherical and of radii r and R and the line joining their centres in position of equilibrium is vertical. Show that the equilibrium is stable provided 6

$$\frac{1}{h}>\frac{1}{r}+\frac{1}{R}$$

where h is the height of the c.g. of the upper body in position of equilibrium above the point of contact.

(d) If a heavy body rests on a fixed body, find the nature of equilibrium. 6

(e) A straight smooth tube revolves with constant angular velocity ω in a horizontal plane about one extremity which is fixed. If initially a particle inside it be at a distance a from a fixed end and moving with constant velocity V along the tube, then show that its distance at time t is 6

$$a\cosh\omega t+\frac{V}{\omega}\sinh\omega t\,.$$

(f) Three forces P, Q, R act along the sides of a triangle formed by the lines $x+y=3$, $2x+y=1$ and $x-y+1=0$. Find the equation of line of action of the resultant. 6

GROUP-C

3. Answer any ***two*** questions: 12×2 = 24

(a) (i) Three forces act along the straight lines $x=0$, $y-z=a$; $y=0$, $z-x=a$; $z=0$, $x-y=a$. Show that they can not reduce to a couple. Prove also that if the system reduces to a single force, its line of action must lie in the surface 6

$$x^2+y^2+z^2-2yz-2zx-2xy=a^2\,.$$

(ii) Three equal uniform rods AB, BC, CD are freely joined and placed in a straight line on a smooth table. The rod AB is struck at its ends by a blow which is perpendicular to its length. Find the resulting motion and show that the velocity of the centre AB is 19 times that of CD and its angular velocity 11 times that of CD. 6

(b) (i) A solid homogeneous cone of height h and vertical angle 2α, oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is 6

$$\frac{h}{5}(4+\tan^2\alpha)$$

(ii) The satellite Vanguard was launched at a velocity of 2000 km per hour at an altitude of 640 km. If the burn out velocity of the last stage was parallel to the Earth's surface, calculate the maximum altitude from the Earth's surface that the satellite will reach. Find also the semi-axes of the orbit and the orbital time. 6

(c) (i) A particle moves on the outside of a smooth elliptic cylinder whose axis is horizontal. The major axis of the principal elliptic section is vertical and the eccentricity of the section is e. If the particle starts from rest on the highest generator and moves in a vertical plane it will leave the cylinder at a point whose eccentric angle is given by $e^2\cos^3\phi = 3\cos\phi - 2$. 6

(ii) A smooth solid circular cone, of height h and vertical angle 2α, is at rest with its axis vertical in a horizontal circular hole of radius 'a'. Show that if $16a > 3h\sin 2\alpha$, the equilibrium is stable and there are two other positions of unstable equilibrium and that if $16a < 3h\sin 2\alpha$, the equilibrium is unstable and the position in which the axis is vertical is the only position of equilibrium. 6

(d) (i) Find the C.G. of a hemisphere whose density varies as the distance from a point on its plane edge. 6

(ii) A force P act along the axis of x and other force nP along a generator of the cylinder $x^2 + y^2 = a^2$. Show that the central axis lies on the cylinder 6

$$n^2(nx - z)^2 + (1+n^2)y^2 = n^4a^2 .$$

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