'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 6th Semester Examination, 2022

CC13-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-II

Time Allotted: 2 Hours — Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

Answer any *four* questions from the following — $3\times 4 = 12$

1. Find all the prime ideals in the ring $\mathbb{Z}_8$.

2. Express the ideal $4\mathbb{Z} + 10\mathbb{Z}$ in the ring $\mathbb{Z}$ as a principal ideal of $\mathbb{Z}$.

3. Show that $1-i$ is irreducible in $\mathbb{Z}[i]$.

4. Give an example of a matrix $A \in M_2(\mathbb{R})$ such that A has no eigenvalue.

5. Test for the diagonalizability of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ in $M_2(\mathbb{R})$.

6. If S_1 and S_2 are two subsets of a vector space V such that $S_1 \subseteq S_2$ then prove that $S_2^0 \subseteq S_1^0$. Here S^0 denotes the annihilator of S.

GROUP-B

Answer any *four* questions from the following — $6\times 4 = 24$

7. (a) Show that $I = \{(a, 0) : a \in \mathbb{Z}\}$ is a prime ideal but not a maximal ideal in the ring $\mathbb{Z}\times\mathbb{Z}$. — 3

   (b) Prove that in an integral domain, every prime element is an irreducible element. Is the converse true? Justify your answer. — 3

8. (a) Show that $2+11i$ and $2-7i$ are relatively prime in the integral domain $\mathbb{Z}[i]$. — 3

   (b) Prove that $K[x]$ is a Euclidean domain where K is a field. — 3

9. (a) Let $\mathcal{B}=\{\beta_1, \beta_2, \beta_3\}$ be a basis for $\mathbb{R}^3$, where $\beta_1=(1, 0, -1)$, $\beta_2=(1, 1, 1)$ and $\beta_3=(2, 2, 0)$. Find the dual basis of $\mathcal{B}$. 3

(b) Let W be the subspace of $\mathbb{R}^5$ which is spanned by the vectors $\alpha_1=(2, -2, 3, 4, -1)$, $\alpha_2=(-1, 1, 2, 5, 2)$, $\alpha_3=(0, 0, -1, -2, 3)$ and $\alpha_4=(1, -1, 2, 3, 0)$. Find W^0. 3

10.(a) Let V be a vector space over a field F and $T: V \to V$ be a linear operator. Suppose $\chi_T(t)$ and $m(t)$ are the characteristic polynomial and minimal polynomial of T respectively. Then prove that $m(t)$ divides $\chi_T(t)$. 3

(b) Prove that for all α, β in a Euclidean space V, $\langle\alpha, \beta\rangle=0$ iff $\|\alpha+\beta\|^2=\|\alpha\|^2+\|\beta\|^2$. 3

11.(a) Let V be an inner product space and T be a linear operator on V. Then prove that T is an orthogonal projection iff T has an adjoint T^* and $T^2=T=T^*$. 4

(b) State Bessel's inequality regarding an orthogonal set of nonzero vectors in an inner product space V. 2

12.(a) Apply Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthonormal basis $\mathcal{B}$ for span (S), where $V=\mathbb{R}^3$ and $S=\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$. 4

(b) Let $A \in M_2(\mathbb{R})$, where $A=\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Show that A is diagonalizable. 2

GROUP-C

Answer any *two* questions from the following $12\times 2=24$

13.(a) Let R be an integral domain. Suppose there exists a function $\delta: R\setminus\{0\} \longrightarrow \mathbb{N}_0$ such that for all $a, b \in R\setminus\{0\}$, $\delta(ab) \geq \delta(b)$, where equality holds iff a is a unit. Then prove that R is a factorization domain. 6

(b) If p be a nonzero non-unit element in a PID D, then prove that the following statements are equivalent: 6

(i) p is a prime element in D.

(ii) p is an irreducible element in D.

(iii) $\langle p\rangle$ is a nonzero maximal ideal of D.

(iv) $\langle p\rangle$ is a nonzero prime ideal of D.

14.(a) Prove that the integral domains $\mathbb{Z}[i\sqrt{n}]$ for $n=6, 7, 10$ are factorization domains but not unique factorization domains. 6

(b) Let $V = M_n(\mathbb{R})$ and $B \in V$ be a fixed vector. If T is the linear operator on V defined by $T(A) = AB - BA$ and if f is the trace function, what is $T^t(f)$? Here T^t denotes the transpose of T. 4

(c) Let $\langle\ ,\ \rangle$ be the standard inner product on $\mathbb{R}^2$. Let $\alpha = (1, 2)$ and $\beta = (-1, 1)$. If γ is a vector such that $\langle \alpha, \gamma \rangle = -1$ and $\langle \beta, \gamma \rangle = 3$, find γ. 2

15.(a) Let F be a field and f be the linear functional on F^2, defined by $f(x_1, x_2) = ax_1 + bx_2$. Then find $T^t f$, where $T : F^2 \to F^2$ is a linear operator defined by $T(x_1, x_2) = (x_1 - x_2,\ x_1 + x_2)$ for all $(x_1, x_2) \in F^2$. 4

(b) Find the minimal polynomial of the matrix $A \in M_3(\mathbb{R})$, where 5

$$A = \begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$$

(c) Let T_1 and T_2 be two linear operators on an inner product space V. Then prove that $(T_1 T_2)^* = T_2^* T_1^*$. 3

16.(a) Let V be an n-dimensional inner product space and W be a subspace of V. Then prove that $\dim(V) = \dim(W) + \dim(W^{\perp})$, where $W^{\perp}$ denotes the orthogonal complement of W. 5

(b) Let T be a linear operator on a finite dimensional vector space V and let $f(t)$ be the characteristic polynomial of T. Then prove that $f(T) = T_0$, where T_0 denotes the zero transformation. 4

(c) Let V be a finite dimensional vector space and W be a subspace of V. Then $\dim(W^0) = \dim V - \dim W$. 3

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